# On the Hamiltonian formulation of Yang–Mills gauge theories

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**Abstract**

The Hamiltonian formulation of the theory proposed in [1, 2] is given both in the Hamilton–De Donder and in the Multimomentum Hamiltonian geometrical approaches. $(3+3)$ Yang–Mills gauge theories are dealt with explicitly in order to restate them in terms of Einstein–Cartan like field theories.

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## 1 Introduction

In some of our recent works [1, 2, 3] a new geometrical framework for Yang–Mills field theories and General Relativity in the tetrad–affine formulation has been developed.
The construction of the new geometrical setting started from the observation that even though the Lagrangian densities of the above theories are defined over the first jet–bundle of the configuration space, they only depend of the antisymmetric combination of field derivatives in the space–time indexes. As for the Yang–Mills case, this is the reason for the singularity in the Lagrangian.
The idea consists is considering a suitable quotient of the first jet–bundle, making two sections equivalent when they possess a first order contact with respect to the exterior covariant differentiation, instead of the whole set of derivatives. The fiber coordinates of the resulting quotient bundle are the antisymmetric combinations of the field derivatives that appear in the Lagrangian.

The geometry of the new space has been widely studied, in order to build as many usual geometric structures of the jet–bundle theory as possible, such as contact forms, jet–prolongations of sections, morphisms and vector fields. These are the geometric tools that are needed to implement variational problems in the Poincaré–Cartan formalism. Moreover, particular choices for the fiber coordinates have been shown to be possible: they consist in the components of the strength tensor for Yang–Mills theories and in the torsion and curvature tensors for General Relativity.
This resulted into the elimination of some un–physical degrees of freedom from the theory (represented by un-necessary jet–coordinates) and to even obtain a *regular Lagrangian* theory in the case of Yang–Mills fields.
This last consideration was the thrust that moved us to write the present work: the presence of a regular Lagrangian allows us to write a Hamiltonian version of the theory proposed in [1, 2]. The advantages arising from the present approach, with respect to the already existing formulations, based on singular Lagrangians (compare, for example, with [4, 5]), are striking. In fact, the singularity of the Lagrangian is the source of known drawbacks: the equations are defined on a constraint sub–manifold, there exist multiple Hamitonian forms associated with the same Lagrangian and the equivalence between Euler–Lagrange and Hamilton equations is not a direct consequence of Legendre transform any more.
On the contrary, the situation in the new geometrical framework is simpler and more elegant: the “Lagrangian” space and the phase–space have the same dimension and the Legendre transform is a (local) diffeomorphism. This ensures the direct equivalence of Lagrangian and Hamiltonian formulations, both in the Hamilton–De Donder (section 3) and in the Multimomentum Hamiltonian approach (section 4).
Finally, we devoted the last section to study the peculiar $(3+3)$ Yang–Mills theory. Starting from the work made in [7, 8], we showed that a coordinate transformation in the phase–space, together with the Poincaré–Cartan approach in our new formalism, allows to describe a (3+3) Yang–Mills theory by means of Einstein–Cartan like equations in 3 dimensions. In particular, in the case of a free $(3+3)$ Yang–Mills field the geometrical construction gives rise to a sort of first–order purely frame–formulation of a General Relativity like theory.
This result is interesting for its further developments: in fact we will show in a subsequent work that an analogous geometrical machinery may be applied to build a *first–order* purely frame–formulation of General Relativity in four dimensions [10].

## 2 The geometrical framework

The present section is devoted to revising the geometrical structure that has been introduced in [1, 2] to describe Yang–Mills theories.

Let $\pi : P \to M$ be a principal fiber bundle, with structural group $G$ and let

$x^i, g^\mu$ denote a system of local fibered coordinates on $P$. $J_1(P)$ denotes the first jet-bundle of $\pi : P \to M$ and it is referred to local coordinates $x^i, g^\mu, g^\mu_i \left(\simeq \frac{\partial g^\mu}{\partial x^i}\right)$.

The space of principal connections on $P$ is identified with the quotient bundle $E := J_1(P)/G$ with respect to the (jet-prolongation of) the right action $R_h$ of the structural group on $P$. If $V^\mu_\nu(g,h)$ represents the differential of the right multiplication $R_h$ in $g \in G$, a set of local coordinates in the quotient space is provided by $x^\mu, a^\mu_i = -g^\nu_i V^\mu_\nu(g, g^{-1})$, subject to the following transformation laws:

$$\bar{x}^i = \bar{x}^i(x^j), \qquad \bar{a}^\mu_i = \left[Ad(\gamma^{-1})^\mu_\nu a^\nu_j + W^\mu_\nu(\gamma^{-1},\gamma)\frac{\partial\gamma^\nu}{\partial x^j}\right]\frac{\partial x^j}{\partial \bar{x}^i} \tag{2.1}$$

where $Ad^\mu_\nu$ and $W^\mu_\nu$ denote respectively the adjoint representation of $G$ and the differential of the left multiplication in $G$, while $\gamma : U \subset M \to G$ ($U$ open set) is an arbitrary smooth map.

As a consequence, the bundle $E \to M$ has the nature of an affine bundle, whose sections represent principal connections over $P \to M$. In fact, every section $\omega : M \to J_1(P)/G$ yields a connection 1-form on $P$, locally described as:

$$\omega(x,g) = \omega^\mu(x,g) \otimes \underline{\mathrm{e}}_\mu := \left[Ad(g^{-1})^\mu_\nu a^\nu_i(x)\, dx^i + W^\mu_\nu(g^{-1},g)\; dg^\nu\right] \otimes \underline{\mathrm{e}}_\mu \tag{2.2}$$

where $\underline{\mathrm{e}}_\mu$ $(\mu = 1, \ldots, r)$ indicate a basis of the Lie algebra $\mathfrak{g}$ of $G$.

Finally, let $\hat{\pi} : J_1(E) \to E$ be the first jet–bundle associated with the bundle $E \to M$, described by the set of local coordinates $x^i, a^\mu_i, a^\mu_{ij} \left(\simeq \frac{\partial a^\mu_i}{\partial x^j}\right)$.

In order to provide a better geometrical framework to describe Yang-Mills gauge theories, the following equivalence relation is introduced in $J_1(E)$: let $\omega_1 = (x^\mu, a^\mu_i, a^\mu_{ij}), \omega_2 = (x^\mu, a^\mu_i, \hat{a}^\mu_{ij}) \in J_1(E)$ be such that $\hat{\pi}(\omega_1) = \hat{\pi}(\omega_2)$, then:

$$\omega_1 \sim \omega_2 \quad \Leftrightarrow \quad (a^\mu_{ij} - a^\mu_{ji}) = (\hat{a}^\mu_{ij} - \hat{a}^\mu_{ji}) \tag{2.3}$$

This means that two sections are declared equivalent if their skew–symmetric derivatives are equal. In more geometric terms, being every section of the bundle $E \to M$ represented by a connection 1–form, the first jet–bundle has been constructed assuming that the equivalence between sections having a first–order contact is evaluated through the exterior differentiation (or, equivalently, the covariant exterior differentiation), instead of the whole set of partial derivatives.

Let $\mathcal{J}(E) := J_1(E)/\sim$ denote the quotient bundle with respect to the above defined equivalence relation and $\rho : J_1(E) \to \mathcal{J}(E)$ the canonical (quotient) projection. The bundle $\mathcal{J}(E)$ is endowed with a set of local fibered coordinates $x^i, a^\mu_i, A^\mu_{ij} := \frac{1}{2}\left(a^\mu_{ij} - a^\mu_{ji}\right)$ $(i < j)$, subject to the following transformation laws:

$$\bar{A}^\mu_{ik} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial x^h}{\partial \bar{x}^k}\left[Ad(\gamma^{-1})^\mu_\nu A^\nu_{jh} + \frac{1}{2}\left(\frac{\partial Ad(\gamma^{-1})^\mu_\nu}{\partial x^h}a^\nu_j - \frac{\partial Ad(\gamma^{-1})^\mu_\nu}{\partial x^j}a^\nu_h\right) + \frac{1}{2}\left(\frac{\partial \eta^\mu_j}{\partial x^h} - \frac{\partial \eta^\mu_h}{\partial x^j}\right)\right] \tag{2.4}$$

where $\eta^\mu_j(x) := W^\mu_\nu(\gamma^{-1}(x), \gamma(x))\frac{\partial \gamma^\nu(x)}{\partial x^j}$.

This newly defined geometrical framework is endowed with the most common features provided by a standard jet–bundle structure.

• *$\mathcal{J}$-extension of sections.* Given a section $\sigma : M \to E$ its $\mathcal{J}$-extension is defined as $\mathcal{J}\sigma := \rho \circ J_1\sigma : M \to \mathcal{J}(E)$, namely projecting the standard jet–prolongation to $\mathcal{J}(E)$ by means of the quotient map. Conversely, every section $s : M \to \mathcal{J}(E)$ will be said to be holonomic if there exists a section $\sigma : M \to E$ such that $s = \mathcal{J}\sigma$.

• *Contact forms.* Let us define the following 2-forms on $\mathcal{J}(E)$:

$$\theta^\mu := da^\mu_i \wedge dx^i + A^\mu_{ij} dx^i \wedge dx^j \tag{2.5}$$

They undergo the transformation laws $\bar{\theta}^\mu = Ad(\gamma^{-1})^\mu_\nu \theta^\nu$. The vector bundle which is locally spanned by the 2-forms (2.5) will be called the contact bundle $\mathcal{C}(\mathcal{J}(E))$ and any section $\eta : \mathcal{J}(E) \to \mathcal{C}(\mathcal{J}(E))$ will be called a contact 2-form. Contact forms are such that $s^*(\eta) = 0$ whenever $s : M \to \mathcal{J}(E)$ is holonomic.

• *$\mathcal{J}$-prolongations of morphisms and vector fields.* A generic morphism $\Phi : E \to E$, fibered over $M$, can be raised to a morphism $\mathcal{J}\Phi : \mathcal{J}(E) \to \mathcal{J}(E)$ considering its ordinary jet–prolongation and restricting it to $\mathcal{J}(E)$ through the quotient map, namely:

$$\mathcal{J}\Phi(z) := \rho \circ j_1\Phi(w) \quad \forall\, w \in \rho^{-1}(z)\ ,\ z \in \mathcal{J}(E)$$

As a matter of fact, not every morphism $\Phi : E \to E$ commutes with the quotient map and produces a well defined prolongation (i.e. independent of the choice of the representative in the equivalence class), but it has to satisfy the condition:

$$\rho \circ j_1\Phi(w_1) = \rho \circ j_1\Phi(w_2) \qquad \forall\, w_1, w_2 \in \rho^{-1}(z) \tag{2.6}$$

Referring to [1] for the proof, it is easy to see that the only morphisms satisfying condition (2.5) are necessarily of the form:

$$\begin{cases} y^i = \chi^i(x^j) \\ \\ b^\nu_i = \Phi^\nu_i(x^j, a^\mu_j) = \Gamma^\nu_\mu(x)\dfrac{\partial x^r}{\partial y^i} a^\mu_r + f^\nu_i(x) \end{cases} \tag{2.7}$$

where $\Gamma^\nu_\mu(x)$ and $f^\nu_i(x)$ are arbitrary local functions on $M$. Their $\mathcal{J}-$prolongation is:

$$\begin{cases} y^i = \chi^i(x^k) \\ \\ b^\nu_i = \Gamma^\nu_\mu(x)\dfrac{\partial x^r}{\partial y^i} a^\mu_r + f^\nu_i(x) \\ \\ B^\nu_{ij} = \Gamma^\nu_\mu A^\mu_{ks}\dfrac{\partial x^k}{\partial y^i}\dfrac{\partial x^s}{\partial y^j} + \frac{1}{2}\left[\dfrac{\partial \Gamma^\nu_\mu}{\partial x^k}\left(\dfrac{\partial x^k}{\partial y^j}\dfrac{\partial x^r}{\partial y^i} - \dfrac{\partial x^k}{\partial y^i}\dfrac{\partial x^r}{\partial y^j}\right) a^\mu_r + \dfrac{\partial f^\nu_i}{\partial x^k}\dfrac{\partial x^k}{\partial y^j} - \dfrac{\partial f^\nu_j}{\partial x^k}\dfrac{\partial x^k}{\partial y^i}\right] \end{cases}$$

In a similar way (compare with [1]), it is easy to prove that the only vector fields of the form

$$X = \epsilon^i(x^j)\,\frac{\partial}{\partial x^i} + \left(-\frac{\partial \epsilon^k}{\partial x^q} a^\mu_k + D^\mu_\nu(x^j) a^\nu_q + G^\mu_q(x^j)\right)\,\frac{\partial}{\partial a^\mu_q} \tag{2.8}$$

(where $\epsilon^i(x)$, $D^\mu_\nu(x)$ and $G^\mu_q(x)$ are arbitrary local functions on $M$) can be $\mathcal{J}-$prolonged to vector fields over $\mathcal{J}(E)$ as follows:

$$\mathcal{J}(X)(z) := \rho_{*\rho^{-1}(z)}(j_1(X)) \qquad \forall\, z \in \mathcal{J}(P) \tag{2.9}$$

The resulting vector field has the form:

$$\mathcal{J}(X) = \epsilon^i(x^j)\,\frac{\partial}{\partial x^i} + \left(-\frac{\partial \epsilon^k}{\partial x^q} a^\mu_k + D^\mu_\nu(x^j) a^\nu_q + G^\mu_q(x^j)\right)\,\frac{\partial}{\partial a^\mu_q} + \sum_{i<j} h^\mu_{ij}\,\frac{\partial}{\partial A^\mu_{ij}}$$

where

$$h^\mu_{ij} = \frac{1}{2}\left(\frac{\partial D^\mu_\nu}{\partial x^j} a^\nu_i - \frac{\partial D^\mu_\nu}{\partial x^i} a^\nu_j + \frac{\partial G^\mu_i}{\partial x^j} - \frac{\partial G^\mu_j}{\partial x^i}\right) + D^\mu_\nu A^\nu_{ij} + \left(A^\mu_{ki}\frac{\partial \epsilon^k}{\partial x^j} - A^\mu_{kj}\frac{\partial \epsilon^k}{\partial x^i}\right)$$

Finally, in order to adapt the geometrical framework to the presence of the covariant differentiation induced by connections, it is useful to introduce a set of new fibered local coordinates over $\mathcal{J}(E)$ of the form:

$$x^i = x^i \quad a^\mu_i = a^\mu_i \quad F^\mu_{ij} = 2A^\mu_{ji} + a^\nu_j a^\rho_i C^\mu_{\rho\nu} \tag{2.10}$$

where $C^\mu_{\rho\nu}$ are the structure coefficients of the group $G$. The latter are subject to the following transformations laws:

$$\bar{F}^\mu_{ik} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial x^h}{\partial \bar{x}^k} Ad(\gamma^{-1})^\mu_\nu F^\nu_{jh} \tag{2.11}$$

Using the new coordinates, every Yang–Mills Lagrangian $m$–form can be expressed as

$$L = \mathcal{L}(x^i, a^\mu_i, F^\mu_{ij})\; ds \tag{2.12}$$

Moreover, it is possible to define a corresponding Poincaré–Cartan $m$-form over $\mathcal{J}(E)$, expressed as

$$\Theta_L := \mathcal{L}\; ds - \frac{1}{2}\theta^\mu \wedge P_\mu \tag{2.13}$$

where $P_\mu := \frac{\partial \mathcal{L}}{\partial F^\mu_{ij}}\, ds_{ij}$, $ds_{ij} := \frac{\partial}{\partial x^i} \rfloor \frac{\partial}{\partial x^j} \rfloor ds$.

The presence of the Poincarè–Cartan form allows to deduce the evolutions equations for Yang–Mills fields looking for the stationary points of the functional

$$A_L(\gamma) := \int_D \gamma^*(\Theta_L) \quad \forall\, \gamma : D \subset M \to \mathcal{J}(E) \tag{2.14}$$

The stationarity condition for $A_L$ (taking null variations at the boundary of the compact domain $D$) is equivalent to the conditions (compare with [1, 2]):

$$\gamma^*(\theta^\mu) = 0 \tag{2.15a}$$

$$\gamma^*\left(\frac{\partial\mathcal{L}}{\partial a_i^\mu} - D_j\frac{\partial\mathcal{L}}{\partial F_{ji}^\mu}\right) = 0 \tag{2.15b}$$

The first equation ensures the kinematic admissibility of the critical section $\gamma$, while the second represents the field equations of the problem. As a matter of fact, the kinematical admissibility is directly obtained from the variational principle and is not imposed as an a-priori condition, as a consequence of the *regularity* of the Lagrangian $\mathcal{L}$ within the new framework provided by $\mathcal{J}(E)$.

# 3 The Hamiltonian framework

Let $\Lambda^m(E)$ denote the modulus of $m$-forms over $E$, and let $\Lambda_r^m(E) \subset \Lambda^m(E)$ ($r < m$) be the sub-bundle consisting of those $m$-forms on $E$ vanishing when $r$ of its given arguments are vertical vectors over the bundle $E \to M$. It is obvious that the above defined bundles form a chain of vector bundles over $E$ such that:

$$0 \subset \Lambda_1^m(E) \subset \Lambda_2^m(E) \subset \ldots \subset \Lambda_r^m(E) \subset \ldots \subset \Lambda^m(E)$$

In particular the attention will be focussed on the first two sub-spaces. Given a system of local coordinates over $E$ and let $ds = dx^1 \wedge \ldots \wedge dx^m$, they can be respectively described as:

$$\Lambda_1^m(E) := \{\omega \in \Lambda^m(E) : \omega = p\, ds\} \tag{3.1}$$

and

$$\Lambda_2^m(E) := \{\omega \in \Lambda^m(E) : \omega = p\, ds + \Pi_\mu^{ji}\, da_i^\mu \wedge ds_j\} \tag{3.2}$$

where $ds_j := \frac{\partial}{\partial x^j} \lrcorner\, ds$. It is then possible to assume $\{x^i, a_i^\mu, p\}$ as a system of local coordinates on $\Lambda_1^m(E)$, subject to the transformations laws $p = J\bar{p}$ (where $J := \det \left\|\frac{\partial \bar{x}^i}{\partial x^k}\right\|$).

A set of local coordinates for $\Lambda_2^m(E)$ is provided by the functions $\{x^i, a_i^\mu, p, \Pi_\mu^{ij}\}$. The latter are subject to a set of transformation laws described by eqs. (2.1), together with:

$$p = J\left(\bar{p} + \bar{\Pi}_\mu^{ji}\left(\frac{\partial Ad(\gamma^{-1})_\nu^\mu}{\partial x^q}a_p^\nu + \frac{\partial \eta_p^\mu}{\partial x^q}\right)\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^i} + \bar{\Pi}_\mu^{ij}(Ad(\gamma^{-1})_\nu^\mu a_p^\nu + \eta_p^\mu)\frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i}\right) \tag{3.3a}$$

$$\Pi_\nu^{pq} = \bar{\Pi}_\mu^{ij} Ad(\gamma^{-1})_\nu^\mu \frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^i} J \tag{3.3b}$$

The bundle $\Lambda_2^m(E)$ is endowed with the canonical Liouville $m$-form, locally expressed as

$$\Theta := p\, ds + \Pi^{ji}_{\mu}\, da^{\mu}_{i} \wedge ds_j \tag{3.4}$$

whose differential

$$\Omega := d\Theta = dp \wedge ds + d\Pi^{ji}_{\mu} \wedge da^{\mu}_{i} \wedge ds_j \tag{3.5}$$

is a multisymplectic $(m+1)-$form over $\Lambda_2^m(E)$.
A deeper geometrical insight in the problem can be given observing that eqs. (3.3) make $\Lambda_1^m$ into a vector sub-bundle of $\Lambda_2^m$, thus allowing us to introduce the quotient bundle $\Lambda_2^m/\Lambda_1^m$. As a consequence of the definition, the latter has the nature of a vector bundle over $E$ and is locally described by the system of coordinates $x^i, a^{\mu}_i, \Pi^{ij}_{\mu}$. It is worth noticing that the transformation law (3.3a) makes $\pi : \Lambda_2^m(E) \to \Lambda_2^m/\Lambda_1^m$ into an affine bundle.
The *phase space* is defined as the vector sub–bundle $\Pi(E) \subset \Lambda_2^m(E)/\Lambda_1^m(E)$ consisting of those elements $z \in \Lambda_2^m(E)/\Lambda_1^m(E)$ satisfying the requirement

$$\Pi^{ij}_{\mu}(z) = -\Pi^{ji}_{\mu}(z) \tag{3.6}$$

Condition (3.6) is well–posed because of the transformation laws (3.3). A local system of coordinates for $\Pi(E)$ is provided by $x^i, a^{\mu}_i, \Pi^{ij}_{\mu}(i<j)$, subject to the same transformation laws (3.3b). Besides, being $\Pi(E)$ a vector sub–bundle, the immersion $i : \Pi(E) \to \Lambda_2^m(E)/\Lambda_1^m(E)$ is well defined and is locally represented by eq. (3.6) itself.
The pull–back bundle $\hat{\pi} : \mathcal{H}(E) \to \Pi(E)$ defined by the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}(E) & \xrightarrow{\hat{i}} & \Lambda_2^m(E) \\ {\scriptstyle\hat{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \Pi(E) & \xrightarrow{i} & \Lambda_2^m(E)/\Lambda_1^m(E) \end{array} \tag{3.7}$$

will now be taken into account. A local coordinate system for $\mathcal{H}(E)$ is provided by $x^i, a^{\mu}_i, \Pi^{ij}_{\mu}(i<j), p$, subject to transformation laws (3.3b), together with:

$$p = J\left(\bar{p} + \bar{\Pi}^{ji}_{\mu}\left(\frac{\partial Ad(\gamma^{-1})^{\mu}_{\nu}}{\partial x^q} a^{\nu}_p + \frac{\partial \eta^{\mu}_p}{\partial x^q}\right)\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^i}\right) \tag{3.8}$$

the latter being the antisymmetric part of eq. (3.3a). The above transformation law shows that the bundle $\hat{\pi} : \mathcal{H}(E) \to \Pi(E)$ has the nature of an affine bundle over the phase space. Every section $h : \Pi(E) \to \mathcal{H}(E)$ will be called a *Hamiltonian section*, and will be locally described in the form:

$$h : p = -\mathcal{H}(x^i, a^{\mu}_i, \Pi^{ij}_{\mu}) \tag{3.9}$$

The presence of the immersion $\hat{\imath} : \mathcal{H}(E) \to \Lambda_2^m(E)$, endows $\mathcal{H}(E)$ with the canonical $m$-form $\hat{\imath}^*(\Theta)$, locally expressed as in eq. (3.4). The latter will be simply denoted as $\Theta$ and will be called the Liouville form on $\mathcal{H}(E)$.
The presence of the $m-$form $\Theta$ on $\mathcal{H}(E)$, allows to create a correspondence between the Hamiltonian and the Lagrangian viewpoints, based on the existence of a unique diffeomorphism $\lambda : \mathcal{J}(E) \to \mathcal{H}(E)$ fibered over $E$ satisfying the requirement:

$$\Theta_L = \lambda^*(\Theta) \tag{3.10}$$

Such a diffeomorphism will be called the *Legendre map.* Given a set of local coordinates $x^i, a_i^\mu, F_{ij}^\mu (i < j)$ on $\mathcal{J}(E)$ and $x^i, a_i^\mu, \Pi_\mu^{ij} (i < j), p$ on $\mathcal{H}(E)$, and taking eqs. (2.10) and (2.13) into account, the Poincaré–Cartan m-form can written as

$$\Theta_L = \left( L - \frac{1}{2} \left( F_{kr}^\mu + a_k^\nu a_r^\rho C_{\rho\nu}^\mu \right) \frac{\partial L}{\partial F_{kr}^\mu} \right) ds + \frac{\partial L}{\partial F_{rk}^\mu} da_k^\mu \wedge ds_r \tag{3.11}$$

and the Legendre map defined by eq. (3.10) is such that:

$$\lambda : \begin{cases} x^i = x^i \\ \\ a_i^\mu = a_i^\mu \\ \\ p(x^j, a_j^\alpha, F_{ij}^\alpha) = L - \frac{1}{2} \left( F_{kr}^\mu + a_k^\nu a_r^\rho C_{\rho\nu}^\mu \right) \frac{\partial L}{\partial F_{kr}^\mu} \\ \\ \Pi_\mu^{ij}(x^j, a_j^\alpha, F_{ij}^\alpha) = \frac{\partial L}{\partial F_{ij}^\mu} \end{cases} \tag{3.12}$$

The most striking feature of the Legendre transformation between $\mathcal{J}(E)$ and $\mathcal{H}(E)$ is provided by its regularity, due to the acquired regularity of the Yang–Mills Lagrangian in the space $\mathcal{J}(E)$. In particular the condition

$$\det \left( \frac{\partial \Pi_\mu^{ij}}{\partial F_{rk}^\alpha} \right) \neq 0 \quad \forall\, i < j \;,\; r < k \quad \forall\, \alpha, \mu$$

assures the local invertibility of the last equation (3.12), allowing to obtain the coordinates $F_{ij}^\alpha$ as functions $F_{ij}^\alpha = F_{ij}^\alpha(x^j, a_j^\alpha, \Pi_\mu^{ij})$. Thus, the Legendre map has the nature of a regular immersion of $\mathcal{J}(E)$ into $\mathcal{H}(E)$, yielding a submanifold $\lambda(\mathcal{J}(E)) \subset \mathcal{H}(E)$, locally described by:

$$p(x^j, a_j^\alpha, \Pi_\mu^{ij}) = L(x^j, a_j^\alpha, \Pi_\mu^{ij}) - \frac{1}{2} \left( F_{kr}^\mu(x^j, a_j^\alpha, \Pi_\mu^{ij}) + a_k^\nu a_r^\rho C_{\rho\nu}^\mu \right) \Pi_\mu^{kr} \tag{3.13}$$

In accordance with the literature, the function

$$H(x^i, a_i^\mu, \Pi_{ij}^\mu) = -L(x^i, a_i^\mu, \Pi_{ij}^\mu) + \frac{1}{2} F_{kr}^\mu(x^i, a_i^\mu, \Pi_\mu^{ij}) \Pi_\mu^{kr} \tag{3.14}$$

will be called the *Hamiltonian* of the system.

If the phase space $\Pi(E)$ is taken into account, the composition $\hat{\lambda} := \hat{\pi} \circ \lambda : \mathcal{J}(E) \to \Pi(E)$ results to be a (local) diffeomorphism. As a consequence, its (local) inverse map $\hat{\lambda}^{-1} : \Pi(E) \to \mathcal{J}(E)$ can be considered. Taking the derivatives of eq. (3.14) with respect to $\Pi^{ij}_\mu$ and using the antisymmetric properties of the coordinates, one gets the coordinate representation for the inverse Legendre map as:

$$\hat{\lambda}^{-1} : \begin{cases} x^i = x^i \\ \\ a^\mu_i = a^\mu_i \\ \\ F^\mu_{ij} = \frac{\partial H}{\partial \Pi^{ij}_\mu} \end{cases} \tag{3.15}$$

Taking the Legendre map into account, as well as its inverse (3.15), it is easy to see that the image $\lambda(\mathcal{J}(E))$ defined by eq. (3.13) yields a Hamiltonian section $h$, represented by a function $\mathcal{H}(x^i, a^\mu_i, \Pi^{ij}_\mu) = H(x^i, a^\mu_i, \Pi^{ij}_\mu) + \frac{1}{2} a^\nu_k a^\rho_r C^\mu_{\rho\nu} \Pi^{kr}_\mu$.
Now, the presence of the Hamiltonian section allows to perform the pull-back of the Liouville form on $\mathcal{H}(E)$ to the phase space $\Pi(E)$. The result is a Hamiltonian dependent $m-$form

$$\Theta_h := h^*(\Theta) = -H(x^i, a^\mu_i, \Pi^{ij}_\mu)\, ds - \Pi^{ij}_\mu \left( da^\mu_i \wedge ds_j + \frac{1}{2} a^\nu_i a^\rho_j C^\mu_{\rho\nu} ds \right) \tag{3.16}$$

The variational principle constructed on the phase space $\Pi(E)$ with the $m$-form $\Theta_h$ yields the Hamilton equations for the problem. In fact, the solution of the variational problem for the functional

$$A_h(\gamma) = \int_D \gamma^*(\Theta_h) \qquad \forall \text{ section } \gamma : \mathrm{D} \subset \mathrm{M} \to \mathcal{H}(\mathrm{E})$$

is made of its Euler–Lagrange equations

$$\gamma^*(X \rfloor d\Theta_h) = 0 \quad \forall\, X \in V(\Pi(E), M) \tag{3.17}$$

where $V(\Pi(E), M)$ denotes the bundle of vectors over $\Pi(E)$ that are vertical with respect to the fibration over $M$.
A straightforward calculation shows that eq. (3.17) splits into the following set of equations:

$$-\frac{\partial H}{\partial \Pi^{ij}_\mu} - \frac{\partial a^\mu_i}{\partial x^j} + \frac{\partial a^\mu_j}{\partial x^i} - a^\nu_i a^\rho_j C^\mu_{\rho\nu} = 0 \tag{3.18a}$$

$$-\frac{\partial H}{\partial a^\mu_i} - \frac{\partial \Pi^{ji}_\mu}{\partial x^j} + \Pi^{ji}_\lambda a^\gamma_j C^\lambda_{\gamma\mu} = 0 \tag{3.18b}$$

usually referred to as *Hamilton–De Donder equations.*

The inverse Legendre map (3.15) shows that eq. (3.18a) is the holonomy requirement for the solution, namely:

$$F^\mu_{ij} = +\frac{\partial a^\mu_j}{\partial x^i} - \frac{\partial a^\mu_i}{\partial x^j} - a^\nu_i a^\rho_j C^\mu_{\rho\nu}$$

On the other hand, eq. (3.18b) can be written in terms of the covariant derivative $D_j$ induced by the connection, giving rise to the usual evolution equations for the Yang–Mills fields:

$$D_j \Pi^{ji}_{\mu} = -\frac{\partial H}{\partial a^{\mu}_{i}}$$

# 4 Multimomentum Hamiltonian formulation

In the previous section a Hamiltonian approach to Yang–Mills field theories has been developed, adapting the already known Hamilton–De Donder formalism developed within the framework of calculus of variations to the new geometrical setting.

Nevertheless, there exists another well–known Hamiltonian approach to field theory, represented by the so–called *multimomentum Hamiltonian formalism*, where Hamiltonian connections play the same role as Hamiltonian vector fields in symplectic geometry.

The argument has been widely studied in the literature (compare with [4, 5]), both on the first jet–bundle and on the Legendre bundle (the phase space), in the Lagrangian and Hamiltonian framework. In this section we will show that a multimomentum formulation of the above theory can be built, starting from the Poincarè–Cartan forms (2.13) and (3.16).

We will start extending some definitions and some results about the Legendre bundle of a generic field theory to our space. All the argument will be presented without proofs; the reader is referred to [4, 5] for comments and further developments.

First of all, the canonical monomorphism is introduced as:

$$\Theta : \Pi(E) \hookrightarrow \Lambda^{m+1} T^*(E) \otimes_M T(M)$$

$$\Theta := -\Pi^{ji}_{\mu}\, da^{\mu}_{i} \wedge ds \otimes \frac{\partial}{\partial x^j} \tag{4.1}$$

The following definitions are strictly associated with monomorphism (4.1).

**Definition 4.1** *The pull–back valued horizontal form* $\Theta$*, locally described by eq.* (4.1) *is called multimomentum Liouville form on the phase space* $\Pi(E)$*.*

**Definition 4.2** *The pull–back valued form, defined as*

$$\Omega := d\Pi^{ji}_{\mu} \wedge da^{\mu}_{i} \wedge ds \otimes \frac{\partial}{\partial x^j} \tag{4.2}$$

*will be called the multisymplectic form on* $\Pi(E)$*.*

The relation between the forms (4.1) and (4.2) is described by the following

**Proposition 4.1** *Given a generic 1–form $\sigma \in \Lambda^1(M)$, the forms* (4.1) *and* (4.2) *are such that*

$$\Omega \rfloor \sigma = -d(\Theta \rfloor \sigma) \tag{4.3}$$

Let us consider a connection $\gamma$ of the bundle $\Pi(E) \to M$, locally described by the tangent–valued horizontal 1–form

$$\gamma = \left( \frac{\partial}{\partial x^k} + \Gamma^{\mu}_{kh} \frac{\partial}{\partial a^{\mu}_h} + \frac{1}{2} \Gamma^{st}_{k\mu} \frac{\partial}{\partial \Pi^{st}_{\mu}} \right) \otimes dx^k \tag{4.4}$$

where $\Gamma^{st}_{k\mu} = -\Gamma^{ts}_{k\mu}$.
Then, the following definition can be given:

**Definition 4.3** *A connection $\gamma$ of the bundle $\Pi(E) \to M$, described by eq.* (4.4), *is called a Hamiltonian connection if the $(m+1)$-form $\gamma \rfloor \Omega$ is closed.*

A straightforward calculation shows that a connection $\gamma$ is Hamiltonian if and only if it satisfies the following conditions:

$$\begin{cases} \dfrac{\partial \Gamma^{ji}_{j\sigma}}{\partial a^{\lambda}_p} = \dfrac{\partial \Gamma^{jp}_{j\lambda}}{\partial a^{\sigma}_i} = 0 \\[2ex] \dfrac{\partial \Gamma^{ji}_{j\sigma}}{\partial \Pi^{pq}_{\lambda}} + \dfrac{\partial \Gamma^{\lambda}_{pq}}{\partial a^{\sigma}_i} - \dfrac{\partial \Gamma^{\lambda}_{qp}}{\partial a^{\sigma}_i} = 0 \\[2ex] \dfrac{\partial \Gamma^{\sigma}_{ji}}{\partial \Pi^{pq}_{\lambda}} - \dfrac{\partial \Gamma^{\lambda}_{pq}}{\partial \Pi^{ji}_{\sigma}} - \dfrac{\partial \Gamma^{\sigma}_{ij}}{\partial \Pi^{pq}_{\lambda}} - \dfrac{\partial \Gamma^{\lambda}_{qp}}{\partial \Pi^{ji}_{\sigma}} = 0 \end{cases} \tag{4.5}$$

**Definition 4.4** *An $m$–form $\eta \in \Lambda^1(\Pi(E))$ is called a multimomentum Hamiltonian form if for every open set $U \subset \Pi(E)$ there exists a Hamiltonian connection on $U$ satisfying the equation*

$$\gamma \rfloor \Omega = d\eta \tag{4.6}$$

Now, we will show that the Poincarè–Cartan form (3.16) is a multimomentum Hamiltonian form. In other words, we will show the existence of Hamiltonian connections $\gamma$ satisfying the equation

$$\gamma \rfloor \Omega = d\Theta_h \tag{4.7}$$

Moreover such connections will be shown to automatically satisfy the Hamilton–De Donder equations (3.18).
As a matter of fact, given a connection $\gamma$ in the form (4.4), we have that

$$\gamma \rfloor \Omega = \Gamma^{ji}_{j\sigma}\, da^{\sigma}_i \wedge ds - \Gamma^{\sigma}_{ji}\, d\Pi^{ji}_{\sigma} \wedge ds + d\Pi^{ji}_{\sigma} \wedge da^{\sigma}_i \wedge ds_j \tag{4.8}$$

Nevertheless, one also as

$$\begin{aligned} d\Theta_h = -\frac{\partial H}{\partial a^{\sigma}_i}\, da^{\sigma}_i \wedge ds - \frac{1}{2}\frac{\partial H}{\partial \Pi^{ji}_{\sigma}}\, d\Pi^{ji}_{\sigma} \wedge ds + d\Pi^{ji}_{\sigma} \wedge da^{\sigma}_i \wedge ds_j + \\ + \frac{1}{2} a^{\nu}_i a^{\rho}_j C^{\sigma}_{\rho\nu}\, d\Pi^{ji}_{\sigma} \wedge ds + \Pi^{ji}_{\mu} C^{\mu}_{\rho\sigma} a^{\rho}_j\, da^{\sigma}_i \wedge ds \end{aligned} \tag{4.9}$$

A direct comparison of eqs. (4.8) and (4.9) gives the algebraic expressions satisfied by the components of $\gamma$:

$$\Gamma^{ji}_{j\sigma} + \frac{\partial H}{\partial a^\sigma_i} - \Pi^{ji}_\mu C^\mu_{\rho\sigma} a^\rho_j = 0 \tag{4.10a}$$

$$\Gamma^\sigma_{ij} - \Gamma^\sigma_{ji} + \frac{\partial H}{\partial \Pi^{ji}_\sigma} - a^\nu_i a^\rho_j C^\sigma_{\rho\nu} = 0 \tag{4.10b}$$

Another direct comparison immediately shows that every integral section of a connection $\gamma$ satisfying eqs. (4.10) automatically verify the Hamilton–De Donder eqs. (3.18).

The Lagrangian version of the above multimomentum Hamiltonian formulation is obtained by means of Legendre transform. In fact the Lagrangian multisymplectic form on $\mathcal{J}(E)$ is defined through the Legendre map as:

$$\Omega_L := d\left(\frac{\partial L}{\partial F^\mu_{ji}}\right) \wedge da^\mu_i \wedge ds \otimes \frac{\partial}{\partial x^j} \tag{4.11}$$

The target connections of the fibration $\mathcal{J}(E) \to M$ are of the form

$$\gamma = \left(\frac{\partial}{\partial x^k} + \Gamma^\mu_{kh} \frac{\partial}{\partial a^\mu_h} + \frac{1}{2}\Gamma^\mu_{kst} \frac{\partial}{\partial F^\mu_{st}}\right) \otimes dx^k \tag{4.12}$$

with $\Gamma^\mu_{kst} = -\Gamma^\mu_{kts}$, and satisfy the equation

$$\gamma \rfloor \Omega_L = d\Theta_L \tag{4.13}$$

Because of the following relation

$$\begin{aligned} d\Theta_L = \frac{\partial L}{\partial a^\mu_i} da^\mu_i \wedge ds - \frac{1}{2} F^\mu_{ij}\, d\left(\frac{\partial L}{\partial F^\mu_{ij}}\right) \wedge ds - d\left(\frac{\partial L}{\partial F^\mu_{ij}}\right) \wedge da^\mu_i \wedge ds_j + \\ -\frac{1}{2} a^\nu_i a^\rho_j C^\mu_{\rho\nu}\, d\left(\frac{\partial L}{\partial F^\mu_{ij}}\right) \wedge ds - \frac{\partial L}{\partial F^\mu_{ij}} C^\mu_{\rho\sigma} a^\rho_j\, da^\sigma_i \wedge ds \end{aligned}$$

it is easily seen that every $\gamma$ solution of eq. (4.13) satisfies the following conditions:

$$\left(\frac{\partial^2 L}{\partial x^j \partial F^\sigma_{ji}} + \Gamma^\mu_{jh} \frac{\partial^2 L}{\partial a^\mu_k \partial F^\sigma_{ji}} + \frac{1}{2}\Gamma^\mu_{jst} \frac{\partial^2 L}{\partial F^\mu_{st} \partial F^\sigma_{ji}}\right) - \frac{\partial L}{\partial F^\mu_{ji}} a^\gamma_j C^\mu_{\gamma\sigma} - \frac{\partial L}{\partial a^\sigma_i} = 0 \tag{4.14a}$$

$$F^\mu_{ij} + \Gamma^\mu_{ji} - \Gamma^\mu_{ij} + a^\lambda_i a^\gamma_j C^\mu_{\gamma\lambda} = 0 \tag{4.14b}$$

Once again, it is easy to verify that the integral sections of such a connection $\gamma$ automatically satisfy Euler–Lagrange equations (2.15).

## 5 $3+3$ Yang–Mills fields

In this section a particular Gauge theory is considered: the base manifold $M$ will be taken to be 3-dimensional and the gauge groups can be equivalently chosen between $G = SO(3)$ and $G = SO(2,1)$.
Under these hypotheses, both space–time and algebra indexes run from 1 to 3. Besides, given a basis $\{\underline{e}_\mu\}$ for the Lie algebra $\mathfrak{g}$ of $G$, we denote by $K_{\mu\nu}$ the coefficients of an Ad-invariant metric over $\mathfrak{g}$ such that the structure coefficients ${C^\mu}_{\lambda\sigma}$ are expressed in the form

$$
{C^\mu}_{\lambda\sigma} = \frac{1}{2}\sqrt{K} K^{\mu\nu} \epsilon_{\nu\lambda\sigma} \tag{5.1}
$$

where $\sqrt{K} = \sqrt{|\det K_{\mu\nu}|}$, $K^{\mu\nu}K_{\nu\sigma} = \delta^\mu_\sigma$ and $\epsilon_{\nu\lambda\sigma}$ are the 3-dimensional Levi–Civita permutation symbols.
The use of a *dual formulation* [6] allows to express such a $(3+3)$ gauge theory in terms of a gravity–like theory in purely metric formulation, as proved in [7, 8].
Now, making an explicit use of the Poincaré–Cartan approach of section 3, we will show that the Hamiltonian version of a $(3+3)$ gauge theory has the same shape as an Einstein–Cartan theory. Borrowing from [7, 8], the central idea consists in performing a local coordinate transformation in the phase space $\Pi(E)$, locally described by the following relations:

$$
\begin{cases} e^\nu_p = \frac{1}{2} K^{\mu\nu} \Pi^{ij}_\mu \epsilon_{pij} \\ \\ \omega_{i\beta\alpha} = \frac{1}{2}\sqrt{K} \epsilon_{\mu\alpha\beta} a^\mu_i \end{cases} \tag{5.2}
$$

where $\epsilon$ denotes the usual 3-dimensional Levi–Civita permutation symbol. The inverse transformation of (5.2) is given by

$$
\begin{cases} \Pi^{ij}_\mu = K_{\mu\nu} e^\nu_p \epsilon^{pij} \\ \\ a^\mu_i = \frac{1}{\sqrt{K}} \epsilon^{\mu\sigma\lambda} \omega_{i\lambda\sigma} \end{cases} \tag{5.3}
$$

It will soon be clear that the coordinates $e^\mu_i$ play the role of the triad coordinates, while the coordinates $\omega_{i\beta\alpha}$ represent the coefficients of the spin–connection.
It is now easy to see that the Poincaré–Cartan 1–form (3.16) in the new coordinates has the form

$$
\Theta_h = -H ds - K_{\mu\nu} e^\nu_p \epsilon^{pij} \frac{1}{\sqrt{K}} \left( \epsilon^{\mu\sigma\lambda} d\omega_{i\lambda\sigma} \wedge ds_j + \frac{1}{2} \epsilon^{\sigma\alpha\beta} {\omega_j}^\mu{}_\sigma \omega_{i\beta\alpha} ds \right) \tag{5.4}
$$

where ${\omega_j}^\mu{}_\sigma := \omega_{j\nu\sigma} K^{\mu\nu}$.

**Proposition 5.1** *The following identities hold identically:*

$$
-\frac{1}{2} \epsilon^{\rho\alpha\beta} \omega_{j\nu\rho} \omega_{i\beta\alpha} = K_{\mu\nu} \epsilon^{\mu\sigma\lambda} \omega_{i\lambda\eta} {\omega_j}^\eta{}_\sigma \tag{5.5}
$$

Proof. A direct calculation shows that the left hand side is such that

$$-\frac{1}{2}\epsilon^{\rho\alpha\beta}\omega_{j\nu\rho}\omega_{i\beta\alpha} = \omega_{j\nu 1}\omega_{i23} - \omega_{j\nu 2}\omega_{i13} + \omega_{j\nu 3}\omega_{i12}$$

while the right hand side becomes

$$\begin{aligned}
K_{\mu\nu}\epsilon^{\mu\sigma\lambda}\omega_{i\lambda\eta}\omega_j{}^\eta{}_\sigma = K_{\nu 1}\omega_{i3\eta}\omega_j{}^\eta{}_2 - K_{\nu 2}\omega_{i3\eta}\omega_j{}^\eta{}_1 + K_{\nu 3}\omega_{i2\eta}\omega_j{}^\eta{}_1 + \\
-K_{\nu 1}\omega_{i2\eta}\omega_j{}^\eta{}_3 + K_{\nu 2}\omega_{i1\eta}\omega_j{}^\eta{}_3 - K_{\nu 3}\omega_{i1\eta}\omega_j{}^\eta{}_2 = \\
K_{\nu 1}\omega_{i31}\omega_j{}^1{}_2 + K_{\nu 1}\omega_{i32}\omega_j{}^2{}_2 - K_{\nu 2}\omega_{i31}\omega_j{}^1{}_1 - K_{\nu 2}\omega_{i32}\omega_j{}^2{}_1 + \\
K_{\nu 3}\omega_{i21}\omega_j{}^1{}_1 + K_{\nu 3}\omega_{i23}\omega_j{}^3{}_1 - K_{\nu 1}\omega_{i21}\omega_j{}^1{}_3 - K_{\nu 1}\omega_{i23}\omega_j{}^3{}_3 + \\
K_{\nu 2}\omega_{i12}\omega_j{}^2{}_3 + K_{\nu 2}\omega_{i13}\omega_j{}^3{}_3 - K_{\nu 3}\omega_{i12}\omega_j{}^2{}_2 - K_{\nu 3}\omega_{i13}\omega_j{}^3{}_2
\end{aligned}$$

Now we notice that:

$$\omega_{j\nu 1}\omega_{i23} = K_{\nu 1}\omega_{i23}\omega_j{}^1{}_1 + K_{\nu 2}\omega_{i23}\omega_j{}^2{}_1 + K_{\nu 3}\omega_{i23}\omega_j{}^3{}_1$$

$$\omega_{j\nu 2}\omega_{i13} = K_{\nu 1}\omega_{i13}\omega_j{}^1{}_3 + K_{\nu 2}\omega_{i13}\omega_j{}^2{}_3 + K_{\nu 3}\omega_{i13}\omega_j{}^3{}_3$$

$$\omega_{j\nu 3}\omega_{i12} = K_{\nu 1}\omega_{i12}\omega_j{}^1{}_2 + K_{\nu 2}\omega_{i12}\omega_j{}^2{}_2 + K_{\nu 3}\omega_{i12}\omega_j{}^3{}_2$$

whence:

$$\begin{aligned}
K_{\mu\nu}\epsilon^{\mu\sigma\lambda}\omega_{i\lambda\eta}\omega_j{}^\eta{}_\sigma = -\frac{1}{2}\epsilon^{\rho\alpha\beta}\omega_{j\nu\rho}\omega_{i\beta\alpha} + \\
- K_{\nu 1}\omega_{i23}\omega_j{}^\mu{}_\mu - K_{\nu 2}\omega_{i31}\omega_j{}^\mu{}_\mu - K_{\nu 3}\omega_{i12}\omega_j{}^\mu{}_\mu
\end{aligned}$$

The conclusion follows from the trace properties of the coefficients $\omega_j{}^\mu{}_\mu = 0$. $\square$

Taking the identity (5.5) into account, we can write the differential of $\Theta_h$ in the form:

$$\begin{aligned}
d\Theta_h = &-\frac{\partial H}{\partial e_i^\lambda}\, de_i^\lambda \wedge ds - \frac{1}{2}\frac{\partial H}{\partial \omega_{i\lambda\sigma}}\, d\omega_{i\lambda\sigma} \wedge ds + \frac{1}{\sqrt{K}}\epsilon^{pij}\epsilon^{\rho\alpha\beta}\omega_{j\rho\nu}e_p^\nu\, d\omega_{i\beta\alpha} \wedge ds \\
&- K_{\mu\nu}\epsilon^{pij}\, de_p^\nu \wedge \frac{1}{\sqrt{K}}\epsilon^{\mu\sigma\lambda}\left(d\omega_{i\lambda\sigma} \wedge ds_j - \omega_{i\lambda\eta}\omega_j{}^\eta{}_\sigma\, ds\right)
\end{aligned} \tag{5.6}$$

Now, let $X = X_p^\nu \frac{\partial}{\partial e_p^\nu} + \frac{1}{2}X_{i\lambda\sigma}\frac{\partial}{\partial \omega_{i\lambda\sigma}}$ be a vertical vector field, with respect to the fibration $\Pi(E) \to M$, on the phase space $\Pi(E)$. We calculate the inner product

$$\begin{aligned}
X \rfloor d\Theta_h = &\left(-\frac{\partial H}{\partial e_p^\nu}\, ds - \epsilon^{pij}\epsilon^{\mu\sigma\lambda}\frac{K_{\mu\nu}}{\sqrt{K}}\, d\omega_{i\lambda\sigma} \wedge ds_j + \epsilon^{pij}\epsilon^{\mu\sigma\lambda}\frac{K_{\mu\nu}}{\sqrt{K}}\omega_{i\lambda\eta}\omega_j{}^\eta{}_\sigma\, ds\right) X_p^\nu \\
&+ \left(-\frac{1}{2}\frac{\partial H}{\partial \omega_{i\lambda\sigma}}\, ds + \epsilon^{pij}\epsilon^{\mu\sigma\lambda}\frac{K_{\mu\nu}}{\sqrt{K}}\, de_p^\nu \wedge ds_j + \frac{1}{\sqrt{K}}\epsilon^{pij}\epsilon^{\rho\sigma\lambda}\omega_{j\rho\nu}e_p^\nu\, ds\right) X_{i\lambda\sigma}
\end{aligned} \tag{5.7}$$

The imposition on the Hamilton–De Donder conditions $\gamma^*(X \rfloor\, d\Theta_h) = 0\ \forall\ X$ yields the final equations

$$-\frac{\partial H}{\partial e^\lambda_i} - \epsilon^{pij}\epsilon^{\mu\sigma\lambda}\frac{K_{\mu\nu}}{\sqrt{K}}\left(\frac{\partial \omega_{i\lambda\sigma}}{\partial x^j} + \omega_{j\lambda\eta}\omega_i{}^\eta{}_\sigma\right) = 0 \tag{5.8a}$$

$$-\frac{\partial H}{\partial \omega_{i\lambda\sigma}} + \frac{2K_{\mu\nu}}{\sqrt{K}}\epsilon^{pij}\epsilon^{\mu\sigma\lambda}\left(\frac{\partial e^\nu_p}{\partial x^j} + \omega_j{}^\nu{}_\gamma e^\gamma_p\right) = 0 \tag{5.8b}$$

representing the Hamilton–De Donder equations in the new coordinates.
As it was anticipated at the beginning of the section, eqs. (5.8) have the form of the 3-dimensional Einstein–Cartan equations, where the coordinates $e^\mu_i$ and $\omega_{i\mu\nu}$ respectively represent the triad components (whenever $\det\|e^\mu_i\| \neq 0$) and the spin–connection coefficients.
In particular, let us consider a free Yang–Mills field, whose dynamical properties are described by the usual Lagrangian density $L = -\frac{1}{4}F^\mu_{ip}F^\nu_{jq}g^{ij}g^{pq}K_{\mu\nu}\sqrt{g}$, where $g_{ij}$ is a given metric over $M$ and $g := |\det g_{ij}|$. Under such circumstances, the Legendre transformation and the Hamiltonian are respectively described by the following equation:

$$\Pi^{ij}_\mu = \frac{\partial L}{\partial F^\mu_{ij}} = -F^\nu_{pq}g^{ip}g^{jq}K_{\mu\nu}\sqrt{g} \quad , \quad H = -\frac{1}{4}\frac{1}{\sqrt{g}}\Pi^{pq}_\sigma\Pi^{st}_\lambda g_{sp}g_{tq}K^{\sigma\lambda}$$

When the new coordinates (5.2) are introduced, the Hamiltonian takes the form:

$$H = -\frac{1}{2}G_{kh}g^{kh}\sqrt{g}\sigma(g) \qquad (G_{hk} := e^\mu_k e^\nu_h K_{\mu\nu})$$

with $\sigma(g)$ representing the sign of $\det\|g_{ij}\|$. Since

$$\frac{\partial H}{\partial \omega_{i\lambda\sigma}} = 0 \quad ; \quad \frac{\partial H}{\partial e^\nu_p} = -e^\mu_k K_{\mu\nu}g^{kp}\sqrt{g}\sigma(g)$$

eqs. (5.8) take the form

$$2K_{\mu\nu}\epsilon^{pij}\epsilon^{\mu\sigma\lambda}\left(\frac{\partial e^\nu_p}{\partial x^j} + \omega_j{}^\nu{}_\gamma e^\gamma_p\right) = 0 \tag{5.9a}$$

$$\frac{1}{2}e^\mu_k K_{\mu\nu}g^{kp}\sqrt{g}\sigma(g) - \epsilon^{pij}\epsilon_{\nu\lambda\sigma}R_{ij}{}^{\lambda\sigma}\sqrt{K}\sigma(K) = 0 \tag{5.9b}$$

where

$$R_{ij\lambda\sigma} = \frac{\partial \omega_{j\lambda\sigma}}{\partial x^i} - \frac{\partial \omega_{i\lambda\sigma}}{\partial x^j} + \omega_{i\lambda\eta}\omega_j{}^\eta{}_\sigma - \omega_{j\lambda\eta}\omega_i{}^\eta{}_\sigma \quad , \quad R_{ij}{}^{\lambda\sigma} = R_{ij\mu\nu}K^{\mu\lambda}K^{\nu\sigma}$$

and $\sigma(K) = \mathrm{sign}(\det\|\mathrm{K}_{\mu\nu}\|)$.
Under the hypothesis $\det\|e^\mu_i\| \neq 0$ eqs. (5.9) have the same form as Einstein equations in the triad–affine formulation. Because of eq. (5.9a), the solution $\omega_{i\mu\nu}(x)$ is

equal to the (spin–connection associated with) Levi–Civita connection induced by the metric $G = K_{\mu\nu} e^\mu(x) \otimes e^\nu(x)$, which is a solution of eq. (5.9b).
More in particular, eqs. (5.9) actually describe a first–order purely frame–formulation of a General Relativity like theory in three dimensions.
Infact, we notice that the transformation laws of the coordinates (5.2) are

$$\bar{e}^\mu_j = e^\sigma_i Ad(\gamma^{-1})^\mu_\sigma \frac{\partial x^i}{\partial \bar{x}^j} \tag{5.10a}$$

and

$$\bar{\omega}_{i\mu\nu} = Ad(\gamma)^\sigma_\mu Ad(\gamma)^\gamma_\nu \frac{\partial x^j}{\partial \bar{x}^i} \omega_{j\sigma\gamma} + Ad(\gamma)^\eta_\mu \frac{\partial Ad(\gamma^{-1})^\sigma_\eta}{\partial x^h} \frac{\partial x^h}{\partial \bar{x}^i} K_{\sigma\nu} \tag{5.10b}$$

Eqs. (5.10a) are the transition functions of a bundle $\pi : \mathcal{T} \to M$, associated with $P \times_M L(M)$ ($L(M)$ being the frame bundle over $M$) through the left action

$$\lambda : (G \times GL(3, \Re)) \times GL(3, \Re) \to GL(3, \Re), \quad \lambda(g, J; X) := Ad(g) \cdot X \cdot J^{-1} \tag{5.11}$$

The (local) sections $e : M \to \mathcal{T}$ may be identified with (local) triads $e^\mu_i(x)\, dx^i$ on $M$; the latter are truly gauge natural objects [9], sensitive to the changes of trivialization of the structure bundle $P$. Each triad $e^\mu$ induces a metric on $M$ expressed as $G := K_{\mu\nu} e^\mu \otimes e^\nu$, which is invariant under transformations (5.10a) by construction.
A new $\mathcal{J}$-bundle $\hat{\pi} : \mathcal{J}(\mathcal{T}) \to M$ can also be constructed by quotienting the first–jet bundle $j_1(\mathcal{T})$ of $\pi : \mathcal{T} \to M$ with respect to an equivalence relation analogous to (2.3). The bundle $\mathcal{J}(\mathcal{T})$ is naturally referred to local coordinates $x^i, e^\mu_i, E^\mu_{ij} := \frac{1}{2}\left(e^\mu_{ij} - e^\mu_{ji}\right)$ $(i < j)$.
Now the idea is to choose the components of the *spin–connections* generated by the triads themselves as fiber coordinates on the bundle $\mathcal{J}(\mathcal{T})$.
Within this framework, let $z = (x^i, e^\mu_i, E^\mu_{ij})$ be an element of $\mathcal{J}(\mathcal{T})$, $x = \hat{\pi}(z)$ its projection over $M$, $e^\mu$ a representative triad belonging to the equivalence class $z$ and $G = K_{\mu\nu} e^\mu \otimes e^\nu$ the metric on $M$ induced by the triad $e^\mu$; we also denote by $\Gamma^k_{ih}$ the Levi–Civita connection induced by the metric $G$ and by $\omega_i{}^\mu{}_\nu$ the spin connection associated with $\Gamma^k_{ih}$ through the triad $e^\mu$ itself.
The relation between the coefficients $\Gamma^k_{ih}$ and $\omega_i{}^\mu{}_\nu$, evaluated in the point $x = \hat{\pi}(z) \in M$, is expressed by the equation

$$\omega_i{}^\mu{}_\nu(x) = e^\mu_k(x) \left( \Gamma^k_{ij} e^j_\nu(x) + \frac{\partial e^k_\nu(x)}{\partial x^i} \right) \tag{5.12}$$

If the coefficients $\Gamma^k_{ih}$ are written in terms of the triad $e^\mu$ and its derivatives, one gets the well–known expression

$$\omega_i{}^\mu{}_\nu(x) := e^\mu_p(x) \left( \Sigma^p{}_{ji}(x) - \Sigma_j{}^p{}_i(x) + \Sigma_{ij}{}^p(x) \right) e^j_\nu(x) \tag{5.13}$$

where

$$\Sigma^p{}_{ji}(x) := e^p_\lambda(x) E^\lambda_{ij}(x) = e^p_\lambda(x) \frac{1}{2} \left( \frac{\partial e^\lambda_i(x)}{\partial x^j} - \frac{\partial e^\lambda_j(x)}{\partial x^i} \right) \tag{5.14}$$

the Latin indexes being lowered and raised by means of the metric $G$. Equations (5.13) and (5.14) show that the values of the coefficients of the spin–connection $\omega_i{}^\mu{}_\nu$, evaluated in $x = \hat{\pi}(z)$, are independent of the choice of the representative $e^\mu$ in the equivalence class $z \in \mathcal{J}(\mathcal{T})$.

Moreover, the torsion–free condition for the connection $\omega_i{}^\mu{}_\nu$ gives a sort of inverse relation of eq. (5.13) in the form

$$2E^\mu_{ij}(x) = \omega_i{}^\mu{}_\nu(x) e^\nu_j(x) - \omega_j{}^\mu{}_\nu(x) e^\nu_i(x) \tag{5.15}$$

Because of the metric compatibility condition $\omega_{i\mu\nu} := \omega_i{}^\sigma{}_\nu K_{\sigma\mu} = -\omega_{i\nu\mu}$, there exists a one-to-one correspondence between the values of the antisymmetric part of the derivatives $E^\mu_{ij}(x) = \frac{1}{2}\left( \frac{\partial e^\mu_i(x)}{\partial x^j} - \frac{\partial e^\mu_j(x)}{\partial x^i} \right)$ and the coefficients of the spin–connection $\omega_{i\mu\nu}(x)$ in the point $x = \hat{\pi}(z)$.

The above considerations allow us to take the quantities $\omega_{i\mu\nu}$ as fiber coordinates of the bundle $\mathcal{J}(\mathcal{T})$, looking at the relations (5.13) and (5.15) as coordinate changes in $\mathcal{J}(\mathcal{T})$.

Finally, it is a straightforward matter to verify that the transformation laws of the spin connection coefficients $\omega_{i\mu\nu}$ coincide with eqs. (5.10b), as well as that the 3-form (5.3) is invariant under coordinate transformations (5.10a), (5.10b).

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